TA: LEE, Yat Long Luca
Email: yllee@math.cuhk.edu.hk
Office: Room 505, AB1
Office Hour: Send me an email first, then we will arrange a meeting (if you need it).

1 Isomorphism

Definition 1.1. Let V and W be vector spaces. Then V is *isomorphic* to W, written $V \cong W$, if there exists a linear transformation $T : V \longrightarrow W$ that is invertible. Such a linear transformation is called an *isomorphism* from V to W.

Useful Facts:

- If V and W are finite-dimensional vector spaces over F. Then V is isomorphic to W if and only if $\dim V = \dim W$.
- (Corollary of the above fact) $V \cong F^n$ if and only if $\dim V = n$.
- Let V and W are finite-dimensional vector spaces over F of dimensions n and m with ordered basis β and γ, respectively. Then the function $\Phi_\beta^\gamma : \mathcal{L}(V, W) \longrightarrow M_{m\times n}(F)$, defined by $\Phi_\beta^\gamma(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$, is an isomorphism.
- $\dim \mathcal{L}(V, W) = mn$.

Exercises

Q1: Get to know what is an isomorphosm

Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(F) \longrightarrow M_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

Q2: First Isomorphism Theorem

This is a standard result, similar statements hold for different algebraic objects, like groups, rings, modules etc. Interested students can read [Abstract Algebra by Dummit & Foote]. I will state the following vector space version:

- Let $T : V \longrightarrow W$ be a linear transformation. Then the linear transformation

  $$\overline{T} : V/\ker T \longrightarrow W$$

  defined by

  $$\overline{T}(v + \ker T) = T(v)$$

  is injective and

  $$V/\ker T \cong \operatorname{im} T.$$

You can try to prove it yourself, or Google for the proof.

2 Change of Coordinate Matrix

You have encountered this many times in calculus.

Idea:

- Given two different ordered bases, we have different representations for the same vector.
- You can think of different bases as "different coordinate axes", then, despite the vector is still geometrically the same, it will have a different representation.
- Given a relation:
$$\begin{cases} x = a_{11}x' + a_{12}y' \\ y = a_{21}x' + a_{22}y' \end{cases}$$
then change of coordinate matrix is simply
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$
One can then generalize the result to higher dimensional cases, that is
- Given $(x_1, ..., x_n)$ is an "old" coordinate w.r.t. β and $(x'_1, ..., x'_n)$ is a "new" coordinate w.r.t β'. Then
$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i$$
where the j-th column of Q is $[x'_j]_\beta$.

The exercises are mainly computational, you may refer to the textbook for some exercises on that.

3 Eigenvalues and Eigenvectors

Definition 3.1. A linear operator T on a finite-dimensional vector space V is called *diagonalizable* if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix A is called *diagonalizable* if L_A is diagonalizable

Remark. $L_A : V \longrightarrow V$ is a linear operator defined by $L_A(x) = Ax$ for some matrix $A \in M_{n\times n}(F)$.

Definition 3.2. Let T be a linear operator on a vector space V. A nonzero vector $v \in V$ is called an *eigenvector* of T if there exists a scalar λ such that $T(v) = \lambda v$. The scalar λ is called the *eigenvalue* corresponding to the eigenvector v.

Useful Facts:

- A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T.
- If T is diagonalizable, $\beta = \{v_1, ..., v_n\}$ is an ordered basis of eigenvectors of T, and $D = [T]_\beta$, then D is a diagonal matrix and D_{ii} is the eigenvalue corresponding to v_i for all $1 \leq i \leq n$.
- An $n \times n$ matrix A is diagonalizable if and only if there exists an ordered basis for F^n consisting of eigenvectors of A. Furthermore, if $\{v_1, ..., v_n\}$ is an ordered basis for F^n consisting of eigenvectors of A and Q is the $n \times n$ matrix whose i-th column is v_i for $i = 1, ..., n$, then $D = Q^{-1}AQ$ is a diagonal matrix such that D_{ii} is the eigenvalue of A corresponding to v_i.
  - In other words, A is diagonalizable if and only if it is similar to a diagonal matrix.

Definition 3.3. Let $A \in M_{n\times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the *characteristic polynomial* of A. If T is a linear operator, then its characteristic polynomial is defined to be $\det([T]_\beta - tI_n)$.

Useful Facts:

- Let $A \in M_{n\times n}(F)$,
  - the characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$;
  - A has at most n distinct eigenvalues.

  Furthermore we let λ be an eigenvalue of A, then

  - a vector $v \in F^n$ is an eigenvector of A corresponding to λ if and only if $v \neq 0$ and $(A - \lambda I)v = 0$.

Exercises

Q3

Source: Textbook §5.1 Q9.

(a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Q4

If T is *nilpotent* linear operator, then the only eigenvalue of T is 0.

Q5

Let $A \in M_{2\times 2}(\mathbb{Z}/p\mathbb{Z})$. Determine the number of characteristic polynomials of A.

Q6

Let A and B be $n \times n$ matrices that are similar, i.e., there exists an invertible $n \times n$ matrix P such that $A = P^{-1}BP$. Show that A and B have the same characteristic polynomial.

4 Diagonalizability

Useful Facts:

- Let T be a linear operator on a vector space, and let $\lambda_1, ..., \lambda_k$ be distinct eigenvalues of T. For each $i = 1, ..., k$, denote S_i be a finite set of eigenvectors of T corresponding to λ_i. If each S_i is linearly independent, then $S_1 \cup \cdots \cup S_k$ is linear independent.
- Let T be a linear operator on an n-dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.

Definition 4.1. A polynomial $f(t)$ over F *splits over* F if there are scalars $c, a_1, ..., a_n$ (not necessarily distinct) in F such that

$$f(t) = c(t - a_1)\cdots(t - a_n)$$

Basic example: Think of $\mathbb{C}$, which every polynomial splits over $\mathbb{C}$ by the fundamental theorem of algebra.

Useful Facts:

- The characteristic polynomial of any diagonalizable linear operator on a vector space V over a field F splits over F.
  - In the sense that $\det([T]_\beta - tI) = \prod_{i=1}^{n}(-1)^n(t - \lambda_i)$.

Definition 4.2. Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The *algebraic multiplicity* of λ is the largest positive integer k for which $(t - \lambda)^k$ is a factor of $f(t)$.
The *geometric multiplicity* of λ is the dimension of its eigenspace.

Note that the two multiplicities may not be the same.
Read textbook §5.1 on "Test for Diagonalizability".

Exercises

Q7

Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) If λ is an eigenvalue of T. Show that the eigenspaces of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) If T is diagonalizable, then T^{-1} is diagonalizable.

Q8: Algebraic Multiplicity and Geometric Multiplicity of eigenvalues

Find all possible Jordan forms for a matrix whose characteristic polynomial is given by

$$(t+2)^2(t-5)^3$$

5 Solution to Exercises

Q1

First of all it is linear, because

$$\Phi(A+C) = B^{-1}(A+C)B = B^{-1}AB + B^{-1}CB = \Phi(A) + \Phi(C)$$

and

$$\Phi(cA) = B^{-1}(cA)B = cB^{-1}AB = c\Phi(A).$$

Next, we let $\Psi(A) := BAB^{-1}$, then

$$\Phi(\Psi(A)) = \Phi(BAB^{-1}) = B^{-1}BAB^{-1}B = A$$

and

$$\Psi(\Phi(A)) = \Psi(B^{-1}AB) = BB^{-1}ABB^{-1} = A$$

hence Ψ is the inverse of Φ.

■

Q2

Google for the solution if you are interested.

Q3

(a) Let T be invertible, then it is bijective, then $T(x) \neq 0$ for all $x \neq 0$. Hence zero is not an eigenvalue of T. Conversely, if T is not invertible, then there must exist some $x \neq 0$ such that $T(x) = 0$, which in this case $\lambda = 0$ is an eigenvalue.

(b) Given that $T(x) = \lambda x$, then $T^{-1}(T(x)) = \lambda T^{-1}(x) \iff \lambda^{-1}x = T^{-1}(x)$.

Q4

If T is nilpotent, i.e., $T^n = 0$ for some integer $n > 0$. Then

$$0 = T^n(x) = T^{n-1}(\lambda x) = T^{n-2}(\lambda^2 x) = \cdots = \lambda^n x$$

hence $\lambda = 0$.

Q5

Write $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

$$\det(A - tI) = (a_{11} - t)(a_{22} - t) - a_{12}a_{21}.$$

For the choice of a_{11} and a_{12}, we have p many options; while for $a_{12}a_{21}$ if $a_{12} = 0$, then the choice of a_{21} does not matter; similarly for $a_{21} = 0$. If $a_{12}, a_{21} \neq 0$, then each of them have $p - 1$ many options.

Q6

$$\begin{aligned} \det(A - tI) &= \det(P^{-1}BP - tP^{-1}IP) \\ &= \det(P^-1(B - tI)P) \\ &= \det(P^{-1}) \det(B - tI) \det P \\ &= \det(B - tI) \end{aligned}$$

Q7

(a) Let $E_{\lambda,T}$ and $E_{\lambda^{-1},T^{-1}}$ denote the eigenspaces of T and T^{-1} with respect to λ and λ^{-1}, respectively.
Let $x \in E_{\lambda,T}$, we have $T(x) = \lambda x$, then $T^{-1}(T(x)) = \lambda T^{-1}(x) \iff \lambda^{-1}x = T^{-1}(x)$. Hence $x \in E_{\lambda^{-1},T^{-1}}$
The other inclusion is done similarly. Hence they are equal.

(b) If T is diagonalizable, then by picking a suitable ordered basis, we have

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Then by (a), we have

$$[T^{-1}]_\beta = ([T]_\beta)^{-1} = [T]_\beta = \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix}.$$

Q8

You can check the book "Algebra" by Michael Artin to learn more about the Jordan form of a matrix. This question is taken out from his book on the chapter about modules.

6 Recording

Link: Here
Password: N=$n0Mz0